INTRODUCTION TO SUPERGRAVITY

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Abstract

Supergravity [1] [2] [3] [4] is a type of quantum theory of elementary particles and their interactions that is based on the particle symmetry known as supersymmetry and that naturally includes gravity along with the other fundamental forces (the electromagnetic force, the weak nuclear force, and the strong nuclear force).

The electromagnetic and the weak forces are now understood to be different facets of a single underlying force that is described by the electroweak theory. Further unification of all four fundamental forces in a single quantum theory is a major goal of theoretical physics. Gravity, however, has proved difficult to treat with any quantum theory that describes the other forces in terms of messenger particles that are exchanged between interacting particles of matter. General relativity, which relates the gravitational force to the curvature of space-time, provides a respectable theory of gravity on a larger scale. To be consistent with general relativity, gravity at the quantum level must be carried by a particle, called the graviton, with an intrinsic angular momentum (spin) of 2 units, unlike the other fundamental forces, whose carriers (e.g., the photon and the gluon) have a spin of 1.

A particle with the properties of the graviton appears naturally in certain theories based on supersymmetry, a symmetry that relates fermions (particles with half-integral values of spin) and bosons (particles with integral values of spin). In these theories supersymmetry is treated as a "local" symmetry; in other words, its transformations vary over space-time, unlike a "global" symmetry, which transforms uniformly over space-time. Treating supersymmetry in this way relates it to general relativity, and so gravity is automatically included. Moreover, these supergravity theories seem to be free from various infinite quantities that usually arise in quantum theories of gravity. This is due to the effects of the additional particles that supersymmetry predicts (every particle must have a supersymmetric partner with the other type of spin). In the simplest form of supergravity, the only

particles that exist are the graviton with spin 2 and its fermionic partner, the gravitino, with spin 3/2. (Neither has yet been observed.) More complicated variants also include particles with spin 1, spin 1/2, and spin 0, all of which are needed to account for the known particles. These variants, however, also predict many more particles than are known at present, and it is difficult to know how to relate the particles in the theory to those that do exist.

Table of Contents

Chapter 1

Introduction

1.1 Review of Supersymmetry

Supersymmetry [5] [6] [7] is a symmetry between boson and fermion fields. The reason why we try to search for such a symmetry because bosons are the mediators of interaction: their statistics allows for a coherent superpositions and thus for a macroscopic force, such as the Coulomb force. On the other hand, fermions are the constituents of matter: their statistics is translated at the macroscopic level into the additive character of matter. Hence it is natural to ask such a fundamental question whether there exists a symmetry which unifies matter and radiation.

In supersymmetry, the supersymmetry charge, Q_r which, by convention, is chosen to be a Majorana spinor. And the basic supersymmetry algebra reads:

$$\{Q_r, \bar{Q}_s\} = 2\gamma^{\mu}_{rs} P_{\mu}$$

$$[Q_r, P^{\mu}] = 0$$

$$[Q_r, M^{\mu\nu}] = i\sigma^{\mu\nu}_{rs} Q_s,$$

where $\bar{Q}_r = (Q^T\gamma^0)_r$ and $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^{\mu}\gamma^{\nu}]$. The last relation simply means that Q_r transform as a spinor in spacetime rotations.

The two basic supermultiplets are (spin 0/spin $\frac{1}{2}$) and (spin 1/spin $\frac{1}{2}$). The former one which is called chiral supermultiplet involves a complex scalar field and a spinor field chosen to be real (Majorana) or to have a given helicity (Weyl). The latter involves a real vector field and a Majorana spinor field and is called vector supermultiplet. If a gauage symmetry is associated. The vector is a gauage field and its supersymmetric partner is called a gaugino.

In Wess-Zumino model, a chiral supermultiplet consists of a complex scalar field $\phi(x) = (A(x) + iB(x))/\sqrt{2}$ and a majorana spinor field $\Psi(x)$. And the number of bosonic and ferminonic degrees of freedom of the chiral supermultiplet are only equal on-shell. Consider a free field Lagrangian

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi + \frac{1}{2}\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi,$$

It is invariant under the infinitesimal supersymmetry transformation

$$\delta A = \bar{\epsilon}\Psi,$$

$$\delta B = i\bar{\epsilon}\gamma_5 \Psi,$$

$$\delta \Psi_r = -[i\gamma^\mu \partial_\mu (A + iB\gamma_5) + m(A + iB\gamma_5)]_{rs}\epsilon_s$$

where ϵ_s is the Majorana spinor parameter of the transformation. But this causes a problem since the algebra of supersymmetry no longer closes off-shell. However we can introduce auxiliary fields, and this provides us with a formulation of supersymmetry where the algebra closes off-shell.

Let us introduce a complex scalar field $F(x) = (F_1(x) + iF_2(x))/\sqrt{2}$ which serves the purpose as an auxiliary field. And consider the Lagarangian

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi + \frac{1}{2}\bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi + \mathcal{L}_{aux}$$

where

$$\mathcal{L}_{aux} = F^* F + m(F\phi + F^* \phi^*).$$

As F has no kinetic term, there is no dynamical degrees of freedom associated with it. And it can be solved by considering its equation of motion $F = -m\phi^*$. Hence one can recovers the on-shell case. In this formulation, the supersymmetry transformations read:

$$\delta A = \bar{\epsilon}\Psi,$$

$$\delta B = i\bar{\epsilon}\gamma_5\Psi,$$

$$\delta\Psi_r = [-i\gamma^\mu\partial_\mu(A + iB\gamma_5) + F_1 - iF_2\gamma_5]\epsilon,$$

$$\delta F_1 = -i\bar{\epsilon}\gamma^\mu\partial_\mu\Psi,$$

$$\delta F_2 = -\bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\Psi.$$

Again we can recover the transformations of the on-shell case if one use the equation of motion $F = -m\phi^*$. When without making use of the equation of motion, in this formulation the supersymmetry algebra closes off-shell. Now we have an off-shell formulation of supersymmetry: the number of off-shell bonsonic degrees of freedom equals the number of off-shell fermionic degrees of freedom when the auxiliary fields are introduced.

To include the case with interactions, we consider the Lagrangian

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi + \frac{1}{2}\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - \lambda(\phi\bar{\Psi}_R\Psi_L + \phi^*\bar{\Psi}_L\Psi_R) + F^*F + F(m\phi + \lambda\phi^2) + F^*(m\phi^* + \lambda\phi^{*2}).$$

And it is invariant under the previous supersymmetry transformations

$$\delta A = \bar{\epsilon}\Psi,$$

$$\delta B = i\bar{\epsilon}\gamma_5\Psi,$$

$$\delta\Psi_r = [-i\gamma^\mu\partial_\mu(A + iB\gamma_5) + F_1 - iF_2\gamma_5]\epsilon,$$

$$\delta F_1 = -i\bar{\epsilon}\gamma^\mu\partial_\mu\Psi,$$

$$\delta F_2 = -\bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\Psi.$$

Solving for $F = -(m\phi^* + \lambda\phi^{*2})$ yields

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi + \frac{1}{2}\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi - \lambda(\phi\bar{\Psi}_R\Psi_L + \phi^*\bar{\Psi}_L\Psi_R) - V(\phi)$$

with $V(\phi) = |m\phi + \lambda\phi^2|^2$. We can introduce the function $W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}\lambda\phi^3$ which is analytic in the field ϕ and is called the superpotential. All interaction terms involve the superpotential and its derivatives

$$\frac{dW}{d\phi} = m\phi + \lambda\phi^2,$$

$$\frac{d^2W}{d\phi^2} = m + 2\lambda\phi.$$

When we are dealing with Standard Model of electroweak interactions, we will use spinors of definite chirality or Weyl spinors. A Weyl spinor has the same on-shell degrees of freedom as Majorana Spinot which is two because out of the four degrees of freedoms of a Dirac spinor, two are projected out by performing the chirality projection. The free field Lagrangain for the chiral supermultiplet with a Weyl Spinor is

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi + \bar{\Psi}_L i\gamma^\mu\partial_\mu\Psi_L - \frac{1}{2}m(\bar{\Psi^c}_R\Psi_L + \bar{\Psi}_L\Psi_R^c) + F^*F + m(F\phi + F^*\phi^*).$$

And it is invariant under the supersymmetry transformation

$$\delta\phi = \sqrt{2}\bar{\epsilon}\Psi_L,$$
$$\delta\Psi_L = \frac{1-\gamma_5}{2}[F - i\gamma^\mu\partial_\mu\phi]\epsilon\sqrt{2},$$
$$\delta F = -i\sqrt{2}\bar{\epsilon}\gamma^\mu\partial_\mu\Psi_L$$

where ϵ is the Majorana spinor of the transformation. This supersymmetry transformation to the one we found in Majorana Spinor, i.e., $\delta A = \bar{\epsilon}\Psi$, $\delta B = i\bar{\epsilon}\gamma_5\Psi$, $\delta\Psi_r = [-i\gamma^\mu\partial_\mu(A + iB\gamma_5) + F_1 - iF_2\gamma_5]\epsilon$, $\delta F_1 = -i\bar{\epsilon}\gamma^\mu\partial_\mu\Psi$ and $\delta F_2 = -\bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\Psi$ except the term $(1-\gamma_5)/2$ appears in the transformation law of the fermion field. And this term is the chirality projector. Its existence is to ensure that it remains left-handed under a supersymmetry transformation. If we take the hermitian conjugate of the above supersymmetry transformation, we get

$$\delta\phi^* = \sqrt{2}\bar{\epsilon}\Psi_R^c,$$
$$\delta\Psi_R^c = \frac{1+\gamma_5}{2}[F^* - i\gamma^\mu\partial_\mu\phi^*]\epsilon\sqrt{2},$$
$$\delta F^* = -i\sqrt{2}\bar{\epsilon}\gamma^\mu\partial_\mu\Psi_R^c$$

where $\Psi_R^c = C(\bar{\Psi_L})^T$.

It can also be easily shown that if we include the interactions to the above free Lagrangian with a special attention to charlities, we will have

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi + \bar{\Psi}_L i\gamma^\mu\partial_\mu\Psi_L - \frac{1}{2}m(\bar{\Psi^c}_R\Psi_L + \bar{\Psi}_L\Psi_R^c) - \lambda(\phi\bar{\Psi}_R^c\Psi_L + \phi^*\bar{\Psi}_L\Psi_R^c) + F^*F$$

$$+F(m\phi+\lambda\phi^2)+F^*(m\phi^*+\lambda\phi^{*2}).$$

For vector supermultiplet, the off-shell formulation contains a real vector field A_μ, a Majorana spinor λ and a real auxiliary pseudoscalar field D. Hence in this off-shell formulation, we have 3+1 bosonic degrees of freedom and 4 fermionic degrees of freedom. In the on-shell case, we have two bosonic and two ferminoic degrees of freedom since the auxiliary field is no longer independent. The free field Lagrangian is

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\bar{\lambda}i\gamma^\mu\partial_\mu\lambda+\frac{1}{2}D^2$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. And it is invariant under the abelian gauage transformation:

$$\delta_g A_\mu=-\frac{1}{g}\partial_\mu\theta,$$

$$\delta_g\lambda_r=0,$$

$$\delta_g D=0.$$

And it is also invariant under the supersymmetry transformation

$$\delta_s A_\mu=\bar{\epsilon}\gamma_\mu\gamma_5\lambda,$$

$$\delta_s\lambda_r=-D\epsilon+r+\frac{1}{2}(\sigma^{\mu\nu}\gamma_5\epsilon)_rF_{\mu\nu},$$

$$\delta_s D=-i\bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\lambda.$$

To couple a chiral supermultiplet with an abelian gauage supermultiplet, first we consider the Lagrangian

$$\mathcal{L}=D^\mu\phi^*D_\mu\phi+\bar{\Psi}_Li\gamma^\mu D_\mu\Psi_L+F^*F+gq(D\phi^*\phi+\sqrt{2}\bar{\lambda}\Psi_L\phi^*+\sqrt{2}\bar{\lambda}\Psi_R^c\phi)$$

where $D_\mu=\partial_\mu-igqA_\mu$ is the covariant derivative , g is the gauge coupling and q the $U(1)$ charge. And it can be shown that it is invariant under

$$\delta_s A_\mu=\bar{\epsilon}\gamma_\mu\gamma_5\lambda,$$

$$\delta_s\lambda_r=-D\epsilon+r+\frac{1}{2}(\sigma^{\mu\nu}\gamma_5\epsilon)_rF_{\mu\nu},$$

$$\delta_s D = -i\bar{\epsilon}\gamma^\mu\gamma_5\partial_\mu\lambda,$$

and the variation of

$$\delta\phi = \sqrt{2}\bar{\epsilon}\Psi_L,$$

$$\delta\Psi_L = \frac{1-\gamma_5}{2}[F - i\gamma^\mu D_\mu\phi]\epsilon\sqrt{2},$$

$$\delta F = -i\sqrt{2}\bar{\epsilon}\gamma^\mu D_\mu\Psi_L - 2gq\phi\bar{\epsilon}\frac{1+\gamma_5}{2}\lambda.$$

To obtain the supersymmetric theory of a chiral supermultiplet coupled with an abelian gauge supermultiplet, we just simply add this Lagrangian with the free field Lagrangian of the Vector supermultiplet.

For nonabelian gauage theories, we can easily generalise it as follows. Let us consider a gauge group G with coupling constant g and structure constants C^{abc}. The generator of the group satisfies $[t^a, t^b] = iC^{abc}t^c$. Then we can form a gauge supermultiplet with a gaugino λ^a, and a real auxiliary D^a for each gauge vector field A^a_μ. The free field Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}\bar{\lambda}^a i\gamma^\mu D_\mu\lambda^a + \frac{1}{2}D^a D^a$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc}A^b_\mu A^c_\nu$$

and

$$D_\mu\lambda^a = \partial_\mu\lambda^a + gC^{abc}A^b_\mu\lambda^c$$

are, respectively, the covariant field strength and the covariant derivative of the gaugino field.

1.2 Review of General Relativity

Since supersymmetry is a spacetime symmetry, local symmetry necesssary involves gravitation. Local symmetry is, therefore, also referred to as supergravity. In this section, we review general relativity, [8] [9] [10] for the classical theory of gravitation whose supersymmetry extension naturally lead to supergravity.

General relativity requires the law of physics to be the same of any observer, be they in a co-ordinate system which is rotating, accelerating, or whatever. This means that the equations describing the laws of physics take the tensor form. Since the derivative of a scalar function gives us a vector function. It is therefore to ask whether the derivative of a tensor function results in a tensor with rank higher than one. Consider the general co-ordinate transformation of $\frac{\partial V^\mu}{\partial x^\nu}$, where V^μ is a contravariant vector. Under the general co-ordinate transformation, we have

$$\frac{\partial V^\mu}{\partial x^\nu} \rightarrow \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial V^\rho}{\partial x^\sigma} + \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho}V^\rho.$$

The presence of the second term shows that $\frac{\partial V^\mu}{\partial x^\nu}$ does not transform as a tensor. Hence we introduce a covariant derivative,

$$D_\nu V^\mu \equiv \partial_\nu V^\mu + \Gamma^\mu_{\rho\nu} V^\rho$$

where $\Gamma^\mu_{\rho\nu}$ is a connection field. Unlike ordinary derivatives, covariant derivatives do not commute. And we have

$$[D_\mu, D_\nu]V^\rho = R^\rho_{\tau\mu\nu}V^\tau + 2A^\tau_{\mu\nu}D_\tau V^\rho$$

where

$$R^\rho_{\tau\mu\nu} = \partial_\mu \Gamma^\rho_{\tau\nu} - \partial_\nu \Gamma^\rho_{\tau\mu} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\tau\nu} - \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\tau\mu}$$

defines the Riemann curvature tensor, and $A^\tau_{\mu\nu}$ is the torsion tensor.

So far we have no mention of the metric tensor in our discussion of the covariant derivative, the connection or even the curvature tensor. In the standard general relativity, we assume that spacetime is a Riemannian manifold. Riemannian manifolds, which are manifolds on which a metric can be defined, form a natural setting for formulating general relativity. On such a manifold, the differential line element is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu.$$

The metric tensors can be used to raise and lower indices in general relativity. To obtain the field equations of general relativity from an action principle, we can try to find an appropiate Lagrangian density, and vary the corresponding action $S = \int \mathcal{L} d^4x$. And the Lagrangian density for the gravitational field is

$$\mathcal{L} = -\frac{1}{2\kappa^2}\sqrt{-g}R$$

where κ^{-2} has the dimension of mass squared and R is the Ricci scalar. By using Palatini formalism wherein the connection fields and their derivatives are regarded as independent fields along with $g_{\mu\nu}(x)$, it leads to Einstein's field equations in a vacuum,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0.$$

So far the previous formulation of general relativity can admit fields transforming as scalars, vectors, and tensors. In supersymmetry, we must necessarily include spinor fields as well, but there exist no generalisation of Spinorial Lorentz transformation rules to general co-ordinate transformations: mathematically speaking, the group $GL(4)$ has no finite dimensional spinor representations. What is done instead yo define, for every point on the curve spacetime, a tangent space with a flat Minkowski metric on which the spinor may transform. Thus the action we construct should be invariant under general co-ordinate transformations on the curved manifold, and invariant under the local Lorentz transformation on the flat tangent space. To do so, We would like to introduce the a basis of orthonormal vectors at each point in spacetime, $e^a_\mu(x)$, which we call the vielbein. The index refers to indices on the spacetime manifold (which is curved in general), while the index labels the different basis vectors. These objects allow us to convert from manifold coordinates to tangent space coordinates. In particular, we can go from the curved-space indices of a warped spacetime to flat-space indices that spinors understand. The choice of an orthonormal basis of tangent vectors means that

$$e^a_\mu(x)e_{a\nu}(x) = g_{\mu\nu}$$

where the a index is raised and lowered with the flat space (Minkowski) metric. In this sense the vielbeins can be thought of as 'square roots of the metric that relate flat and curved coordinates. Moreover, we could have arbitrarily defined the tangent space z-direction pointing in one direction or another relative to the manifold'ss basis so long as the two directions are related by a Lorentz transformation. Thus we have an symmetry (or whatever symmetry applies to the manifold). Further, we could have made this arbitrary choice independently for each point in spacetime. This means that the symmetry is local, i.e. it is a gauge symmetry. Like any other gauge symmetry, we are required to introduce a gauge field for the Lorentz group, which we shall call $\omega^{a}_{\mu\nu}(x)$.From the point of view of Riemannian geometry this is just the connection, so we can alternately call it the spin connection. In particular, any vector field with manifold indices, $V^{\mu}(x)$,can now be recast as a vector field with tangent-space indices, i.e., $V^{a} = e^{a}_{\mu}(x)V^{\mu}(x)$. Note that the covariant derivative is defined as usual for multi-index objects: a partial derivative followed by a connection term for each index. For the manifold index theres a Christoffel connection, while for the tangent space index theres a spin connection:

$$D_{\mu}e^{a}_{\nu}(x) = \partial_{\mu}e^{a}_{\nu} - \Gamma^{\lambda}_{\mu\nu}e^{a}_{\lambda} + \omega^{a}_{\mu b}e^{b}_{\nu}.$$

By requiring that both objects have the same covariant derivative, we get the constraint

$$D_{\mu}e^{a}_{\mu}(x) = 0.$$

And the spin connection fields ω^{ab}_{μ} can be constructed from knowledge of the vierbein via,

$$\omega^{ab}_{\mu} = \frac{1}{2}e^{a\nu}(\partial_{\mu}e^{b}_{\nu} - \partial_{\nu}e^{b}_{\mu} + \frac{1}{4}e^{a\rho}e^{b\sigma}(\partial_{\sigma}e^{c}_{\rho} - \partial_{\rho}e^{c}_{\sigma})e_{c\mu} - (a \leftrightarrow b).$$

1.3 Review of Superspace

In this section, we give a breif review of superspace [11] [12]. An arbitary function of translated or Lorentz transformed corrdinates can be represented by

$$f(x+a) = \exp(ia^{\mu}P_{\mu})f(x) = \exp^{a^{\mu}\partial_{\mu}} f(x),$$

$$f(\Lambda x) = \exp(-\frac{i}{2}\theta^{\mu\nu}L_{\mu\nu})f(x) = \exp^{\frac{1}{2}\theta^{\mu\nu}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})} f(x).$$

We can extend it to the supersymmetry transformations. In other words, we can generalise space adding new coordinates such that a supersymmetry transformation is nothing else than a translation in these new coordinates. Since the supersymmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$ have fermionic anticommuting character, the same will be true for these new coordinates, θ^α and $\bar{\theta}^{\dot{\alpha}}$. We rewrite the generators of supersymmetry transformations with these coordinares and their derivatives. We have

$$Q_\alpha = a\frac{d}{\partial\theta^\alpha} + b(\sigma^\mu\bar{\theta})_\alpha\partial_\mu,$$

$$\bar{Q}_\alpha = \bar{a}\frac{d}{\partial\bar{\theta}^{\dot{\alpha}}} + \bar{b}(\sigma^\mu\theta)_{\dot{\alpha}}\partial_\mu.$$

with $a, b, \bar{a}, \bar{b}$ are complex numbers. A generic field of the coordinates $x^\mu, \theta^\alpha\bar{\theta}^{\dot{\alpha}}$ is called a superfield. A superfield can be expanded in a finite series of ordinary space coordinate functions:

$$F(x,\theta,\bar{\theta}) = f_1(x) + \theta\psi_1(x) + \bar{\theta}_2(x) + \theta\theta f_2(x) + \bar{\theta}\bar{\theta} f_3(x) + \theta\sigma^\mu(x)\bar{\theta}\nu_\mu(x) +$$

$$\theta\theta\bar{\theta}\bar{\psi}_3(x) + \bar{\theta}\bar{\theta}\theta\psi_4(x) + \theta\theta\bar{\theta}\bar{\theta} f_4(x).$$

The superfield has supersymmetry transformation as

$$\delta F(x,\theta,\bar{\theta}) = i(\epsilon Q + \bar{\epsilon}\bar{Q})F(x,\theta,\bar{\theta}).$$

Since such a general superfield has much more components than necessary. So we can introduce an operator $\hat{O}_i$ that reduce the number of degrees of freedom. This operator when acting on the superfield will gives us zero and it satifies

$$\{\hat{O}_i, Q_\alpha\} = \{\hat{O}_i, \bar{Q}_{\dot{\alpha}}\} = 0,$$

$$[\hat{O}_i, Q_\alpha] = [\hat{O}_i, \bar{Q}_{\dot{\alpha}}] = 0.$$

which depends on its fermionic or bosonic character. And in particular, if we define the operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - (\frac{\bar{b}}{\bar{a}})(\sigma^\mu\bar{\theta})_\alpha\partial_\mu,$$

$$\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - (\frac{b}{a})(\theta\sigma^{\mu})_{\dot{\alpha}}\partial_{\mu}.$$

Then the chiral superfields are defined as those which satisfy

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

and the antichiral superfields are those satisfy

$$D_{\alpha}\bar{\Phi} = 0.$$

A general superfield is a function on $x^{\mu}, \theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$ but the chiral superfield is a function of only θ^{α} and the combination of

$$y^{\mu} = x^{mu} - (\frac{b}{a})\theta\sigma^{\mu}\bar{\theta}.$$

And the supersymmetry generators are simplified to

$$Q_{\alpha} \to a\frac{\partial}{\partial \theta^{\alpha}},$$

$$\bar{Q}_{\dot{\alpha}} \to \frac{\bar{a}b + \bar{b}a}{a}(\theta\sigma^{\mu})_{\dot{\alpha}}\partial^{y}_{\mu}.$$

where the spacetime derivative ∂^{y}_{μ} is a derivative w.r.t y coordinates. Chiral superfield can be expanded as

$$\Phi(y, \theta) = A\phi(y) + B\theta\psi(y) + C\theta\theta F(y).$$

And its supersymmetry transformation is

$$A\delta\phi(y) + B\theta\delta\psi(y) + C\theta\theta\delta F(y) =$$

$$i(a\epsilon^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} + \frac{\bar{a}b + \bar{b}a}{a}(\theta\sigma^{\mu}a\bar{\epsilon})\partial^{y}_{\mu})(A\phi(y) + B\theta\psi(y) + C\theta\theta F(y)).$$

where

$$B = \frac{A}{ia},$$

$$C = \frac{A}{2a^2},$$

$$\bar{a}b + \bar{b}a = i.$$

in order to satisfy the infinitesimal supersymmetric transformations. For an antichiral superfield, it is a function of $\bar{\theta}^{\dot{\alpha}}$and the combination of

$$\bar{y}^{\mu} = x^{\mu} + (\frac{\bar{b}}{\bar{a}})\theta\sigma^{\mu}\bar{\theta}.$$

And the supersymmetry generators are

$$\bar{Q}_{\dot{\alpha}} \to \bar{a}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}},$$

$$Q_{\alpha} \to \frac{\bar{a}b + \bar{b}a}{\bar{a}}(\bar{\theta}\sigma^{\mu})_{\alpha}\partial^{\bar{y}}_{\mu}.$$

And antichiral superfield can be expanded as

$$\bar{\Phi}(\bar{y}, \bar{\theta}) = \bar{A}\bar{\phi}(\bar{y}) + \bar{B}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{C}\bar{\theta}\bar{\theta}\bar{F}(\bar{y}).$$

And its supersymmetry transformation is

$$\bar{A}\delta\bar{\phi}(\bar{y}) + \bar{B}\bar{\theta}\delta\bar{\psi}(\bar{y}) + \bar{C}\bar{\theta}\bar{\theta}\delta\bar{F}(\bar{y}) =$$

$$i(\bar{a}\bar{\epsilon}^{\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \frac{\bar{a}b + \bar{b}a}{\bar{a}}\epsilon\sigma^{\mu}\bar{\theta}\partial^{\bar{y}}_{\mu})(\bar{A}\bar{\phi}(\bar{y}) + \bar{B}\bar{\theta}\bar{\psi}(\bar{y}) + \bar{C}\bar{\theta}\bar{\theta}\bar{F}(\bar{y})).$$

where

$$\bar{B} = -\frac{\bar{A}}{i\bar{a}},$$

$$\bar{C} = \frac{\bar{A}}{2\bar{a}^2},$$

$$\bar{a}b + \bar{b}a = i.$$

Requiring that the hermitian conjugate of the chiral superfield is an antichiral superfield, we get

$$\bar{A} = A^*, \bar{a} = a^*.$$

The chiral and antichiral superfields then expanded as

$$\frac{1}{A}\Phi(y, \theta) = \phi(y) + \frac{1}{ia}\theta\psi(y) + \frac{1}{2a^2}\theta\theta F(y),$$

$$\frac{1}{A^*}\bar{\Phi}(\bar{y}, \bar{\theta}) = \bar{\phi}(\bar{y}) - \frac{1}{ia^*}\bar{\theta}\bar{\psi}(\bar{y}) + \frac{1}{2a^{*2}}\bar{\theta}\bar{\theta}\bar{F}(\bar{y}).$$

The transformations of the fields are now simplified as

$$\delta\Phi(y,\theta) = i(a\epsilon^{\alpha}\frac{\partial}{\partial\theta^{\alpha}} + \frac{i}{a}\theta\sigma^{\mu}\bar{\epsilon}\partial_{\mu}^{y})\Phi(y,\theta),$$

$$\delta\bar{\Phi}(\bar{y},\bar{\theta}) = i(a^{*}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\bar{\epsilon}^{-\dot{\alpha}} + \frac{i}{a^{*}}\epsilon\sigma^{\mu}\bar{\theta}\partial_{\mu}^{\bar{y}})\bar{\Phi}(\bar{y},\bar{\theta}).$$

A vector field is nothing but a real superfield, i.e.,

$$[V(x,\theta,\bar{\theta})]^{\dagger} = V(x,\theta,\bar{\theta}).$$

In general, it can be expanded as

$$\begin{gathered}V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \theta\sigma^{\mu}\bar{\theta}\nu_{\mu}(x) \\ +\frac{i}{2}\theta\theta[M(x)+iN(x)] - \frac{i}{2}\bar{\theta}\bar{\theta}[M(x)-iN(x)] \\ +i\theta\theta\bar{\theta}\bar{\theta}[\bar{\lambda}(x) + \frac{i}{2}\partial_{\mu}\chi(x)\sigma^{\mu}] - i\bar{\theta}\bar{\theta}\theta\theta[\lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)] \\ +\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) - \frac{1}{2}\partial^{\mu}\partial_{\mu}C(x)].\end{gathered}$$

And there are too many degrees of freedom. Using the fact that if V is a vector multiplet so is $V+\Phi+\Phi^{\dagger}$, where Φ is a chiral superfield. We can show that the gauge transformations

$$V \rightarrow V_{WZ} = V + \Phi + \Phi^{\dagger}$$

with some chose chiral multiplet Φ can bring this vector multiplet into a simpler form

$$V_{WZ}(x,\theta,\bar{\theta}) = \theta\sigma^{\mu}\bar{\theta}\nu_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\theta\bar{\lambda}(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x).$$

And this is the Wess-Zumino gauge.

To build a Lagrangian which is supersymmetric invariant, we first observe that the highest component (F) of a chiral superfield transforms under supersymmetry translation as a total derivative, so its spacetime integral is a supersymmetry invariant. So the $\theta\theta$ component of the chiral superfield is a supersymmetry invariant (up to total derivatives) and thus a possible invariant term in the Lagrangian:

$$[\Phi]_{\theta\theta}.$$

And also any combinations of the chiral superfields is also a supersymmetric invariant term in the Lagrangian:

$$[\Pi_{i=1}^{n}\Phi_i^{n_i}]_{\theta\theta}.$$

Similarly for the antichirial superfield, it is the compenent term $\bar{\theta}\bar{\theta}$ which is invariant.

$$[\bar{\Phi}]_{\bar{\theta}\bar{\theta}}.$$

And also any combinations of the chiral superfields is also a supersymmetric invariant term in the Lagrangian:

$$[\Pi_{i=1}^{n}\bar{\Phi}_i^{n_i}]_{\bar{\theta}\bar{\theta}}.$$

For vector multiplets, the highest component term transforms as a total derivative. So

$$[V]_{\theta\theta\bar{\theta}\bar{\theta}}$$

is supersymmettic invariant.(up to total derivatives.)And a good candidate for a supersymmetry invariant Lagrangian.A general supersymmetric invariant Lagrangian is

$$\mathcal{L} = [K(\Phi, \Phi^{\dagger})]_{\theta\theta\bar{\theta}\bar{\theta}} + [W(\Phi)]]_{\theta\theta} + [W^{\dagger}(\Phi^{\dagger})]]_{\theta\theta}$$

where the real function $K(\Phi, \Phi^{\dagger})$ called K$\ddot{a}$hler potential and the holomorphic function $W(\Phi)$ is the superpotential. We have also chosen $\bar{W} = W^{\dagger}$ in order to satisfy the hermiticity condition for the Lagrangian. For example we can choose the following K$\ddot{a}$hler potential and $W(\Phi)$ superpotential to construct the free Lagrangian:

$$K(\Phi, \Phi^{\dagger}) = \Phi^{\dagger}\Phi$$

$$W(\Phi) = \frac{m}{2}\Phi^2$$

We can generalise the free Lagrangian above for an arbitrary interacting case. And it is called Wess-Zumino model when without gauge interactions. To derive a general superpotential of a single chiral superfield $W(\Phi)$, we can expand this superpotential around its bosonic component ϕ:

$$W(\Phi) = W(\phi) + \frac{\partial W}{\partial \phi}(\phi)(\Phi - \phi) + \dots$$

which gives

$$[W(\Phi)]_{\theta\theta} = -\frac{\partial W}{\partial \phi}(\phi)F - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}(\phi)\psi\psi.$$

The Lagrangian of a single chiral superfield with

$$K(\Phi, \Phi^\dagger) = \Phi^\dagger \Phi$$

and a general superpotential $W(\Phi)$ gives

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi + \bar{\psi} i \bar{\sigma}^\mu \partial_\mu \psi + F^* F$$

$$-[\frac{\partial W}{\partial \phi}(\phi)F - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}(\phi)\psi\psi + h.c.]$$

Then one can use the equation of motion of the auxiliary field to determine it:

$$F^* = \frac{\partial W}{\partial \phi}$$

and then the single field Wess-Zumino Lagrangian becomes

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - |\frac{\partial W}{\partial \phi}|^2 + \bar{\psi} i \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}\frac{\partial^2 W}{\partial \phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^*}{\partial \phi^{*2}}\bar{\psi}\bar{\psi}.$$

The its potential is

$$V = |F|^2 = |\frac{\partial W}{\partial \phi}|^2.$$

The most general renormalizable single field Wess-Zumino model is

$$W(\Phi) = a\Phi + \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3.$$

And the auxiliary field in this case is

$$F^* = a + m\phi + \lambda\phi^2.$$

Then the Lagrangian in this case is

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - |a + m\phi + \lambda\phi^2|^2$$

$$+\bar{\psi} i \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}(m + 2\lambda\phi)\psi\psi - \frac{1}{2}(m^* + 2\lambda^* \phi^*)\bar{\psi}\bar{\psi}.$$

In order to show that the bosonic and fermionic masses are equal, we expand the bosonic field

$$\phi = \nu + \varphi$$

with

$$a + m\nu + \lambda\nu^2 = 0, < \varphi >= 0.$$

Then the above Lagrangian takes the following form

$$\mathcal{L} = \partial^\mu \varphi^* \partial_\mu \varphi - |\mu\varphi + \lambda\varphi^2|^2$$

$$+\bar{\psi} i\bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}(\mu + 2\lambda\varphi)\psi\psi - \frac{1}{2}(\mu^* + 2\lambda^*\varphi^*)\bar{\psi}\bar{\psi}$$

with

$$\mu = m + 2\lambda\nu$$

which is the common bosonic and femionic mass.

Chapter 2

Supersymmetry and supergravity in a simple model

2.1 Rigid N = 1 Supersymmetry in a simple model

In this chapter, we will review a paper written by Peter van Nieuwenhuizen [13]. In this paper, the author considers a simple model with only one coordinate t instead of 3 + 1 dimensional Minkowski space with coordinates x, y, z and t in order to demonstrate the basic principles behind supergravity. In this model, we consider two point particles in t-space. They are the real bosonic field $\phi(t)$ and the real fermionic field $\lambda(t)$. Both these fields have their space-dependence suppressed. $\phi(t)$ is a smooth of t and its derivatives are well-defined. $\lambda(t)$ is an independent Grassmann number for each fixed t. And so it satisfies $\lambda(t_1)\lambda(t_2) = -\lambda(t_2)\lambda(t_1)$. The Lagrangian in this model takes the following form:

$$\mathcal{L}_R = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda}.$$

where dot represents the time derivative. And the anticommutators between $\lambda(t)$ and $\dot{\lambda}$ are

$$\{\lambda(t), \lambda(t)\} = 0, \{\lambda(t), \dot{\lambda(t)}\} = 0, \{\dot{\lambda(t)}, \dot{\lambda(t)}\} = 0.$$

Since we require the Lagrangian to be hermitian, that is why we have a factor i for the above Lagrangian. The term $\frac{1}{2}\dot{\phi}^2$ is a truncation of the Klein-Gordon action to an x, y, z independent field. And the term $\frac{i}{2}\lambda\dot{\lambda}$ is the truncation of the Dirac action for a real spinor to one of its components.

Since the supersymmetry transformatios transform bosons into fermions, and vice versa, the parameter ϵ must be anticommuting as ϕ is commuting and λ is anticommuting. And at the same time, we want to make sure the dimension of the action to be zero, we claim that the supersymmetry transformatios are:

$$\delta\phi = i\epsilon\lambda,$$

$$\delta\lambda = -\dot{\phi}\epsilon.$$

If we varies the fields in action $S = \int \mathcal{L}_{\mathcal{R}}(t)$ according to the transformation above, we get

$$\delta S = \int [\dot{\epsilon}(i\dot{\phi}\lambda) - \frac{i}{2}\frac{d}{dt}(\lambda\dot{\phi}\epsilon)]dt.$$

If we assume that the fields tend to zero at $t = \pm\infty$ plus the fact that ϵ is a constant, we have $\delta S = 0$.

On ϕ, we can show that

$$[\delta\epsilon_2, \epsilon_1]\phi = (2i\epsilon_1\epsilon_2)\dot{\phi}.$$

And on λ,

$$[\delta\epsilon_2, \epsilon_1]\lambda = (2i\epsilon_1\epsilon_2)\dot{\lambda}.$$

Hence this reveals that the algebra of rigid supersymmetry transformations is a square root of translations. In high dimensional theories, there will be an additional term proportional to the field equation of the fermion in the above commutator on a fermion. Hence one would need to introduce auxiliary fields to cancel this extra term.

2.2 N = 1 supergravity in a simple model

We can distinguish between rigidly supersymmetric field theories, which have a constant symmetry parameter, and locally supersymmetric field theories whose symmetry parameter is an arbitary space-time dependent parameter. As a gauge field is required in the local symmetry, we have a gauge field called gravitino in locally supersymmetric field theories.

And this local symmetry is different from the one in Einstein's theory of gravitation. In Einstein's theory of gravitation, it is from the diffeomorphism invariance, and the gauge field is the metric field $g_{\mu\nu}(X)$. Gauge theories of supersymmetry requires curved spacetime. Hence gravity is needed to construct gauge theories of supersymmetry. And this is the reason the local symmetry is called supergravity. The quanta of the metric $g_{\mu\nu}(X)$ is a massless particle called gravitons. It is the bosonic partner of the gravitino.

We start with the same action S_R as introduced in the previous section. Now we allow ϵ to be time-dependent. Then we have

$$\delta S_R = \int_{-\infty}^{\infty} \dot{\epsilon}(i\dot{\phi}\lambda)dt.$$

In order to cancel this variation, we introduce the gauge field, the gravitino $\psi(t)$. Then we can now couple it the Noether current of the rigid supersymmetry and use

$$\delta\phi = \dot{\epsilon} + \ldots.$$

And the action of this Noether current is

$$S_N = \int_{-\infty}^{\infty} (-i\psi\dot{\phi}\lambda)dt.$$

Now if we vary the ψ in this action of Noether current, the variation of S_R cancels out the variation of S_N. But the fields $\dot{\phi}$ and λ in S_N will be varied too. And it will cause two further variations in S_N And we get

$$\delta S_N = \int_{-\infty}^{\infty} [-i\psi\{\frac{d}{dt}(i\epsilon\lambda)\}\lambda + i\psi\dot{\phi}\epsilon\dot{\phi}]dt.$$

Since $\lambda\lambda = 0$, so we have

$$\delta S_N = \int_{-\infty}^{\infty} i\psi(\dot{\phi}\dot{\phi} + i\lambda\dot{\lambda})\epsilon dt.$$

In order to cancel the last term we can simply add a new term in $\delta\lambda$. But to get rid of the first term, we have to introduce a new field, graviton h and couple it with $\dot{\phi}\dot{\phi}$. And this shows the reason why local susy is a theory of gravity, and hence the name supergravity.

Now we have another choice how to couple this new field h. It can also couple it with $i\lambda\dot{\lambda}$. And for the most general case, we can consider to couple the linear combination of both $\dot{\phi}\dot{\phi}$ and $i\lambda\dot{\lambda}$. So we add

$$S_S = -\int_{-\infty}^{\infty} h[\dot{\phi}\dot{\phi} - i\lambda\dot{\lambda}x]dt.$$

$$\delta h = -i\epsilon\psi,$$

$$\delta\lambda = -i\dot{\psi} + i(1+x)\psi\lambda\epsilon,$$

where x is an arbitary constant real parameter. Now the variation, $\delta h = -i\epsilon\psi$ in this new action $-\int h\dot{\phi}\dot{\phi}dt$ cancels the first term in δS_N. And the new variation $i(1+x)\psi\lambda\epsilon$ in $\frac{i}{2}\lambda\dot{\lambda}$ of $\mathcal{L}_R$ and the new variation $\delta h = -i\epsilon\psi$ in S_S cancel the variation of δS_N.

So far we have got rid of δS_N. But now we have to consider the variation of the matter fields in this new term S_S. We seperate it into two cases. The first case is when $x = 0$ and the second case is when $x \neq 0$.

When $x = 0$, we need only vary the $\dot{\phi}$ in S_S. This gives us

$$\delta\mathcal{L}_S = -2h\dot{\phi}i\dot{\epsilon}\lambda - 2h\dot{\phi}i\epsilon\dot{\lambda}.$$

As the first term is proportional to the Noether current $\dot{\phi}\lambda$ in S_N and so we add a new term in $\delta\psi$ to cancel it. And this new term is

$$\delta(new)\psi = -2h\dot{\epsilon}.$$

The second term in $\delta\mathcal{L}_S$ is proportional to the free field equation of λ can so it can be cancelled by a new term to the transformation law of λ. And this new term is $2h\dot{\phi}\epsilon$.

Now this new term $2h\dot{\phi}\epsilon$ also produces a new variation in the Noether action S_N.

$$\delta\mathcal{L}_N(\delta\lambda = 2h\dot{\phi}\epsilon) = -i\psi\dot{\phi}(2h\dot{\phi}\epsilon).$$

And this term is proportional to $\dot{\phi}\dot{\phi}$, so we add a final term in δh to cancel it. This term is $2ih\epsilon\psi$. Finally we have cancelled all the variations and obtain the following final results:

So the final results are

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda} - i\psi\dot{\phi}\lambda - h\dot{\phi}^2,$$

$$\delta\phi = i\epsilon\lambda,$$

$$\delta\lambda = -\dot{\phi}\epsilon + i\psi\lambda\epsilon + 2h\dot{\phi}\epsilon,$$

$$\delta\psi = \dot{\epsilon} - 2h\dot{\epsilon},$$

$$\delta h = -i\epsilon\psi + 2ih\epsilon\psi.$$

Now we take a look at the susy algebra of this model. On ϕ, we find

$$[\delta(\epsilon_2), \delta(\epsilon_1)]\phi = [2i(1-2h)\epsilon_2\epsilon_1]\dot{\phi} + i[-2i\epsilon_2\epsilon_1\psi]\lambda.$$

There is a general coordinate transformation $\delta\psi = \hat{\xi}\dot{\phi}$ with $\hat{\xi} = 2i(1-2h)\epsilon_2\epsilon_1$ at the right hand side. This is the gravitational extension of the nongravitational rigid transformation with parameter $\xi = 2i\epsilon_2\epsilon_1$ that we found in the rigid susy commutator. And the second term is a local susy transformation $i\hat{\epsilon}\lambda$ of ϕ with parameter $\hat{\epsilon} = -2i\epsilon_2\epsilon_1\psi$. On λ, we have

$$[\delta(\epsilon_2), \delta(\epsilon_1)]\lambda = \hat{\xi}\dot{\lambda} - \dot{\phi}\hat{\epsilon} + 2h\dot{\phi}\hat{\epsilon}.$$

The term $\hat{\xi}$ constitutes a general coordinate transformation on λ. On ψ, we have

$$[\delta(\epsilon_2), \delta(\epsilon_1)]\psi = \frac{d}{dt}\hat{\epsilon} - 2h\frac{d}{dt}\hat{\epsilon} + \hat{\xi}\dot{\psi}.$$

Finally on h, we have

$$[\delta(\epsilon_2), \delta(\epsilon_1)]h = -i\hat{\epsilon}\psi + 2ih\hat{\epsilon}\psi + \hat{\xi}\dot{h} - \dot{\hat{\xi}}h + \frac{1}{2}\dot{\hat{\xi}}.$$

The terms $\hat{\xi}$ constitutes a general coordinates transformation of h. Therefore the local susy algebra closes on all fields uniformly.

For the case $x \neq 0$, we just simply use the rescaling method. We rescale λ and ψ as follows:

$$\lambda = (1+2hx)^{1/2}\tilde{\lambda},$$

$$\psi(1+2hx)^{1/2} = \tilde{\psi}.$$

Hence we have produced the same action as the case $x = 0$,

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}(1+2hx)\tilde{\lambda}\frac{d}{dt}\tilde{\lambda} - i\lambda\psi\dot{\phi}\tilde{\lambda} - h\dot{\phi}^2.$$

It shows that this action is also locally susy. Now we find the susy transformation under this rescaling. First we have

$$\delta\lambda = -\dot{\phi}\epsilon + i\tilde{\psi}\tilde{\lambda}\epsilon + 2h\dot{\phi}\epsilon.$$

Then we divide it by the factor $(1+2hx)^{1/2}$ and use $\delta h = -i\epsilon\psi + 2ih\epsilon\psi$. We get

$$\delta\tilde{\lambda} = -(1-2h)\dot{\phi}\tilde{\epsilon} + i(\frac{1+x}{1+2hx})\tilde{\psi}\tilde{\lambda}\tilde{\epsilon}.$$

For the case $x = -1$, we have

$$\delta\tilde{\lambda} = -(1-2h)\dot{\phi}\tilde{\epsilon},$$

$$\delta\phi = i\tilde{\epsilon}(1-2h)\tilde{\lambda}.$$

We can find δh and $\delta\tilde{\phi}$ in a similar manner.

$$\delta h = (1-2h)[(1-2h)\dot{\tilde{\epsilon}} - \dot{h}\tilde{\epsilon}].$$

Now we apply the Noether method to compute the Lagrangian and the susy transformations directly to gravity. And we will have slighly different results.

We first start with the rigid Lagrangian that we encountered before

$$\mathcal{L}_R = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda}.$$

For local ξ, we consider the translation symmetry rule

$$\delta\psi = \xi\dot{\phi},$$

$$\delta\lambda = \xi\dot{\lambda}.$$

The variation of the action becomes

$$\delta S_R = \int [\dot{\phi}\frac{d}{dt}(\xi\dot{\phi}) + \frac{i}{2}\lambda\frac{d}{dt}(\xi\dot{\lambda}) + \frac{i}{2}\xi\dot{\lambda}\dot{\lambda}]dt.$$

The third term vanishes since $\dot{\lambda}\dot{\lambda} = 0$, and the second terms vanishes too after partial integration. And after the partial integration of the first term, we have

$$\delta S_R = \int [\frac{1}{2}\dot{\xi}\dot{\phi}\dot{\phi} + \frac{d}{dt}(\frac{1}{2}\xi\dot{\phi}\dot{\phi} + \frac{i}{2}\xi\lambda\dot{\lambda})]dt.$$

Now since $\dot{\phi}\dot{\phi}$ is the Noether current for the translations. Hence we introduce the gauge field h for gravity. Hence we obtain

$$\mathcal{L}_N = -h\dot{\phi}\dot{\phi},$$

$$\delta h = \frac{1}{2}\dot{\xi}.$$

We variate $\dot{\phi}$ in $\mathcal{L}_N$ and get

$$\delta S_N = \int -2h\dot{\phi}\frac{d}{dt}(\xi\dot{\phi})dt.$$

And this can be cancelled out by an additional term in δh. And this term is $\xi\dot{h} - h\dot{\xi}$. Hence we have

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda} - h\dot{\phi}^2.$$

And it is invariant under

$$\delta\phi = \xi\dot{\phi},$$

$$\delta\lambda = \xi\dot{\lambda},$$

$$\delta h = \frac{1}{2}\dot{\xi} + \xi\dot{h} - \dot{\xi}h.$$

Now we add the coupling to the gravitino $-i\psi\dot{\phi}\lambda$. So we require that it should also be invariant under the local ξ transformation. And it turns out that $\delta\psi$ should transform as

$$\delta\psi = \xi\dot{\psi}.$$

To see that, we compute $\delta(-i\psi\dot{\phi}\lambda)$. We obtain

$$\delta(-i\psi\dot{\phi}\lambda) = -i\delta\psi\dot{\phi}\lambda - i\psi[\frac{d}{dt}(\xi\dot{\phi})]\lambda - i\psi\dot{\phi}\xi\dot{\lambda}.$$

Perform a partial integration on the last two terms, we get

$$-i\delta\psi\dot{\phi}\lambda + i\dot{\psi}\xi\dot{\phi}\lambda.$$

And these two terms can be cancelled by choosing $\delta\psi = \xi\dot{\psi}$. After rescaling $\lambda = \sqrt{1+2hx}\tilde{\lambda}$ and $\psi\sqrt{1+2hx} = \tilde{\psi}$, and setting x = 1, we have

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda} - h(\dot{\phi}^2 + i\lambda\dot{\lambda}) - i\psi\dot{\phi}\lambda,$$

$$\delta\phi = \xi\dot{\phi},$$

$$\delta h = \frac{1}{2}\dot{\xi} + \xi\dot{h} - \dot{\xi}h,$$

$$\delta\lambda = \xi\dot{\lambda} + \frac{1}{2}\dot{\xi}\lambda,$$

$$\delta\psi = \xi\dot{\psi} - \frac{1}{2}\dot{\xi}\psi.$$

where we have dropped the tildas. And for the susy transformation rules, we have the followings:

$$\delta\phi = i\epsilon(1-2h)\lambda,$$

$$\delta\lambda = -(1-2h)\dot{\phi}\epsilon,$$

$$\delta\psi = (1-2h)[(1-2h)\dot{\epsilon} - \dot{h}\epsilon],$$

$$\delta h = -(1-2h)i\epsilon\psi.$$

This action can also be rewritten as

$$\mathcal{L} = \frac{1}{2}(1-2h)(\dot{\phi}^2 + i\lambda\dot{\lambda}) - i\psi\dot{\phi}\lambda.$$

Chapter 3

Supergravity in 4-d

In this chapter we are going to review the physics reports titled "Supergravity" written by P. van Nieuwenhuizen [14]. Papers related to supergravity in $4-d$ can also be found in [15] [16] [17] .In the last chapter, we have seen that supergravity is the gauge theory of supersymmetry in our toy model. Since we have only considered one dimensional coordinate in this toy model we will genearlise it into the case when we have 4-dimensional space-time coordinates in this chapter.

3.1 Supergravity

Let us denote a scalar, psesudoscalr and spin 1/2 field as A, B and λ, respectively in this chapter. For global symmetry, we have seen from chapter 1 that the Lagrangian is a sum of the Klein-Gordon actions and the Dirac action. We state in here again since we want to follow the same notations as in P. van Nieuwenhuizen's papers.

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\lambda}\not{\partial}\lambda, \bar{\lambda} = \lambda^\dagger\gamma^4.$$

Its globally invariant supersymmetric rules are:

$$\delta A = \frac{1}{2}\bar{\epsilon}\lambda,$$

$$\delta B = -\frac{i}{2}\bar{\epsilon}\gamma_5\lambda,$$

$$\delta\lambda = \frac{1}{2}\not{\partial}(A - iB\gamma_5)\epsilon.$$

To see how supergravity arises, we make ϵ^α local. Then for example the spin $\frac{1}{2}$ field now transforms as

$$\delta\lambda = \frac{1}{2}\partial\!\!\!/(A - iB\gamma_5)\epsilon(x).$$

This transformation does not have the $\partial_\mu\epsilon$ term because λ is a matter field and only a supersymmetry gauge field has this deriative term. When ϵ is a constant, the variation of the action is zero. So for the local ϵ, the variation of the action must be proportional to $\partial_\mu\epsilon$. So we get

$$\delta I = \int d^4x(\partial_\mu\bar{\epsilon}(x))j_N^\mu.$$

where j_N^μ is the Noether current. For constant ϵ, we have

$$\delta\mathcal{L} = \partial_\mu K^\mu$$

where

$$K^\mu = -\frac{1}{4}\bar{\epsilon}\gamma^\mu[\partial\!\!\!/(A - i\gamma_5 B)]\lambda.$$

And in the local case we have

$$\delta\mathcal{L} = \partial_\mu K^\mu + (\partial_\mu\bar{\epsilon}(x))s^\mu$$

for some s^μ. And indeed we can show that s^μ is just the Noether current. Now we can apply the Noether method. So we add the following Noether coupling action:

$$I^N = \int d^4x(-k\bar{\psi}_\mu)j_N^\mu.$$

and requires that gravitino field transforms as

$$\delta\psi_\mu \sim \partial_\mu\epsilon(x) + \ldots.$$

By dimensional analysis, we can see that we require a dimensional coupling k to appear such that

$$\delta\psi_\mu = k^{-1}\partial_\mu\epsilon(x) + \ldots.$$

Now we have three arguments which indicate that local supersymmetry requires gravity. The first argument is from the appearence of the dimensional coupling k. The second argument can be seen from

$$[\delta(\epsilon_1), \delta(\epsilon_2)]B \sim \frac{1}{2}(\bar{\epsilon}_2(x)\gamma^\mu\epsilon_1(x))\partial_\mu B.$$

Hence it means that there is a translation of $\frac{1}{2}\bar{\epsilon}_2(x)\gamma^\mu\epsilon_1(x)$ and it is different from point to point. This is the same as a general coordinate transformation in general relativity. To see the third argument we variate I^N. Then we have new terms due to δA, δB , $\delta\lambda$ in I^N.By considering only the terms quadratic in A and B, we have

$$\delta(I + I^N) = \int d^4x \frac{k}{2}(\bar{\psi}_\mu\gamma_\nu\epsilon)(T^{\mu\nu}(A) + T^{\mu\nu}(B))$$

where

$$T_{\mu\nu}(A) = \partial_\mu A\partial_\nu A - \frac{1}{2}\delta_{\mu\nu}(\partial_\lambda A)^2$$

is the energy momentum tensor of the field A. In order to cancel this term, we add a second Noether coupling with a new field $g_{\mu\nu}$ to the Noether current of translations, $\frac{1}{2}T^{\mu\nu}$ and require that it transforms as

$$\delta g_{\mu\nu} = -\frac{k}{2}\psi_\mu\gamma_\nu\epsilon - \frac{k}{2}\bar{\psi}_\nu\gamma_\mu\epsilon.$$

Since ferminos must present in supersymmetry, hence we use tetrads e^a_μ to describe the gravitational field instead of $g_{\mu\nu}$. And the supersymmetry transformation of e^a_μ is

$$\delta e^m_\mu = \frac{1}{2}k\bar{\epsilon}\psi_\mu.$$

And for the transformation law of the gravtino, we have

$$\delta\psi^a_\mu = k^{-1}D_\mu\epsilon$$

where

$$D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{2}\omega^{mn}_\mu\sigma_{mn}\epsilon$$

since we are in curve space. The gauge fields must also satisfy the global supersymmetry. For the helicity$(2, 3/2)$, its representation of global supersymmetry are

$$\delta g_{\mu\nu} = \frac{k}{2}(\bar{\epsilon}\gamma_\mu\psi_\nu + \bar{\epsilon}\gamma_\nu\psi_\mu),$$

$$\delta\psi_\mu = \frac{1}{2k}(\omega_\mu^{mn})_L\sigma_{mn}\epsilon$$

where L stands for linearised and ϵ is some constant.

3.2 The gauge action of simple supergravity

There are three fields in the gauge action. They are the tetrad e_μ^m, the gravintino ψ_μ^a and the connection ω_μ^{mn}. The ω_μ^{mn} should not be physical because e_μ^m and ψ_μ^a have already formed a boson- fermino doublet with adjacent helicities ± 2 and $\pm\frac{3}{2}$. So we start out with two independent fields e_μ^m and ω_μ^{mn}. Then ω_μ^{mn} is eliminated as an independent field by solving its nonpropagating field equation, and ω_μ^{mn} becomes a function of e_μ^m as a result. And this soultion actually satisfies the tetrad postulate

$$\partial_\mu e_\nu^m + \omega_\mu^{mn}(e)e_{n\nu} - \Gamma_{\nu\mu}^\alpha(g)e_\alpha^m = 0.$$

We will discuss how to obtain the dependence of ω_μ^{mn} on other fields by Palatini formalism in next section. Now we take a look at the bonsonic part of the gauage action of supergravity. It takes the usual hilbert action R Usually we would define the Hilbert action in terms of the connection $\Gamma_{\nu\mu}^\alpha$ as follows

$$R = \delta_\mu^\sigma g^{\nu\rho} R_{\nu\rho\sigma}^\mu(\Gamma)$$

$$R_{\nu\rho\sigma}^\mu = \partial_\rho\Gamma_{\nu\sigma}^\mu - \partial_\sigma\Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\lambda\Gamma_{\nu\rho}^\mu - \Gamma_{\nu\rho}^\lambda\Gamma_{\lambda\sigma}^\mu.$$

But it is more fundamental and meaningful to define the hilbert action by using the spin connection ω_μ^{mn} rather than $\Gamma_{\nu\mu}^\alpha$. And group theory gives us the following curvature

$$R_{\mu\nu}^{mn}(\omega) = \partial_\mu\omega_\nu^{mn} - \partial_\nu\omega_\mu^{mn} + \omega_\mu^{mc}\omega_{\nu c}^{m} - \omega_\nu^{mc}\omega_{\mu c}^{n}.$$

Now we can write down the Hilbert action as

$$\mathcal{L}^{(2)} = -\frac{1}{2k^2}\sqrt{g}R(g,\Gamma) = -\frac{1}{2k^2}eR(e,\omega),$$

$$R(e,\omega) = e^{\mu\nu}e^{n\mu}R_{\mu\nu mn}(\omega)$$

where $e = \det e^m_\mu$. For the ferminonic part of the gauge action, we expect it is linearised and quadratic in ψ^a_μ and contain only one derivative. In order to make sure that it has positive energy, we have the following same action as the Rarita and Schwinger action

$$\mathcal{L} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma.$$

And it is unique up to the shifting by $\lambda\gamma_\sigma\gamma\cdot\phi$ with $\lambda \neq -\frac{1}{4}$, and $\psi_\sigma = \partial_\sigma\epsilon(x)$. And in the curve space, we have the following extension

$$\mathcal{L}^{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma$$

$$D_\rho\psi_\sigma = (\partial_\rho + \frac{1}{2}\omega^{mn}_\rho\sigma_{mn})\psi_\sigma.$$

It can be shown that the Noether method that we used previously in the Wess-Zumino model to gravity can actually give us these gauge actions shown above. In the last section we have arrived the following matter action

$$I + I^N = \int d^4x\frac{e}{2}[(\partial_\mu A\partial_\nu A + \partial_\mu B\partial_\nu B)g^{\mu\nu} + \bar{\lambda}\not{D}(\omega(e))\lambda - k\bar{\psi}_\mu(\not{\partial}(A + i\gamma_5 B))\gamma^\mu\lambda]$$

But we have not considered the term with AB. In the following, we see how the gauge actions arised when we try to cancel this term. Now we vary λ in the Noether term and consider the AB term. We have

$$\delta I + I^N = \int d^4x - \frac{ik}{2}(\bar{\psi}_\mu\gamma_5\gamma_\tau\epsilon^{\rho\sigma\mu\tau})(\gamma_5\epsilon)(\partial_\rho A\partial_\rho B).$$

So we can cancel the above variation by adding either terms to the action or to the transformation laws. First we do a partial integrate $\partial_\rho A$ or $\partial_\rho B$, and interpret $D_\rho(\bar{\psi}_\mu\gamma_5\gamma_\tau)\epsilon^{\rho\sigma\mu\tau}$ as the gravitino field equation. Then by adding to $\delta\psi_\mu$ a term $\frac{ik}{4}\gamma_5\epsilon(A\partial_\mu B)$, it can cancel part of the above variation. And finally th same gravitino action $\mathcal{L}^{3/2} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma$ is arised as before. So far, up to this point, we only obtain the gravitino action with the spin connection $\omega(e)$ but not$\omega(e,\psi)$ but we could find an extra term which turn $\omega(e)$ into $\omega(e,\psi)$ at the next level in k. The Noether method used above by requiring gauge invariace of the matter to obtain gauge action is peculiar to supergravity. It comes from the appearence of the matter fields in the gauge field transformations if one does not use auxiliary fields. And auxiliary fields can be viewed as Lagrange multipliers to get rid of the matter terms.

3.3 Palatini formalism and Flat supergravity with torsion

Now we are going to solve the field equation for the spin connection. And from there, we can find the torsion induced by gravitinos. By using the following identity

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}e^m_\mu e^n_\nu = 2e(e^\rho_c e^\sigma_d - e^\sigma_c e^\rho_d)$$

we can rewrite the Hilbert action as

$$\mathcal{L}^{(2)} = \frac{1}{8k^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}e^m_\mu e^n_\nu R^{cd}_{\rho\sigma}(\omega).$$

After varying the spin connection, we have

$$\delta R^{cd}_{\rho\sigma}(\omega) = D_\rho\delta\omega^{cd}_\sigma - D_\sigma\delta\omega^{cd}_\rho,$$

$$D_\rho\delta\omega^{cd}_\sigma = \partial_\rho\delta\omega^{cd}_\sigma + \omega^{ce}_\rho\delta\omega^d_{\sigma e} + \omega^{de}_\rho\delta\omega^c_{\sigma e}.$$

After partial integration, we have

$$\delta\mathcal{L}^{(2)} = \frac{1}{2k^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{mncd}(D_\sigma e^m_\mu)e^n_\nu\delta\omega^{cd}_\rho$$

$$D_\sigma e^m_\mu = \partial_\sigma e^m_\mu + \omega^{mn}_\sigma e_{n\mu}.$$

We then vary the spin connection in the Rarita-Schwinger action

$$\delta\mathcal{L}^{3/2} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu\gamma_5\gamma_\nu\sigma_{cd}\psi_\sigma)(\delta\omega^{cd}_\rho).$$

By using the fact that

$$\bar{\psi}_\mu \gamma_5 \gamma_\nu \sigma_{cd} \psi_\sigma = \frac{1}{2} \bar{\psi}_\mu \gamma_5 (e_{c\nu} \gamma_d - e_{d\nu} \gamma_c) \psi_\sigma + \frac{1}{2} e^b_\nu \epsilon_{bcdm} \bar{\psi}_\mu \gamma^m \psi_\sigma$$

and $\bar{\psi}_\mu \gamma_5 \gamma_d \psi_\sigma$ is symmetric in μ and σ while $\bar{\psi}_\mu \sigma^a \psi_\sigma$ is antisymmetric. We get

$$\delta \mathcal{L}^{3/2} = -\frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \epsilon_{ncdm} (\bar{\psi}_\mu \gamma^m \psi_\sigma) e^n_\nu (\delta \omega^{cd}_\rho).$$

By comparing the variation of Hilbert and Rarita-Schwinger action above, we have

$$D_\mu e^m_\nu - D_\nu e^m_\mu = \frac{k^2}{2} (\bar{\psi}_\mu \gamma^m \psi_\nu)$$

Now we introduce the contorsion tensor k^{mn}_μ defined by

$$\omega^{mn}_\mu = \omega^{mn}_\mu(e) + k^{mn}_\mu,$$

$$\omega_{\mu mn}(e) = \frac{1}{2} e^\nu_m (\partial_\mu e_{n\nu} - \partial_\nu e_{n\mu}) - \frac{1}{2} e^\nu_n (\partial_\mu e_{m\nu} - \partial_\nu e_{m\mu}) - \frac{1}{2} e^\rho_m e^\sigma_n (\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho}) e^c_\mu.$$

With $D_\mu e^m_\nu - D_\nu e^m_\mu = \frac{k^2}{2} (\bar{\psi}_\mu \gamma^m \psi_\nu)$ and the first tetrad postulate, we have

$$\partial_\mu e^m_\nu + \omega^m_{\mu\nu}(e) - (\mu \leftrightarrow \nu) = 0,$$

$$k_{\mu m\nu} - k_{\nu m\mu} = \frac{k^2}{2} \bar{\psi}_\mu \gamma_m \psi_\nu.$$

Finally by substituting the above equation, we obtain

$$\omega_{\mu mn}(e, \psi) = \omega_{\mu mn}(e) + \frac{k^2}{4} (\bar{\psi}_\mu \gamma_m \psi_\nu - \bar{\psi}_\mu \gamma_n \psi_m + \bar{\psi}_m \gamma_\mu \psi_n).$$

Now we can apply the second tetrad postulate

$$\partial_\mu e^a_\nu + \omega^a_{\mu\nu}(e) - \Gamma^\alpha_{\nu\mu} e^a_\alpha = 0.$$

Then we can get an expression for $\Gamma^\alpha_{\nu\mu}$ from the above equation of $\omega_{\mu mn}$. We define the antisymmetric part of $\Gamma^\alpha_{\nu\mu}$ as the torsion.

$$S^\alpha_{\mu\nu} = \frac{1}{2} (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}).$$

And since

$$\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu} = -k^{\alpha}_{\mu\nu} + k^{\alpha}_{\nu\mu},$$

hence the torsion is

$$S^{\alpha}_{\mu\nu} = -\frac{k^2}{4}\bar{\psi}_{\mu}\gamma_{\alpha}\psi_{\nu}.$$

Finally, by putting all the results together, we have

$$\omega_{\mu mn} = \frac{1}{2}(R_{\mu n,m} - R_{\mu m,n} + R_{mn,\mu})$$

$$R_{\mu\nu,m} = -\partial_{\mu}e_{m\nu} + \partial_{\nu}e_{m\mu} + \frac{k^2}{2}\bar{\psi}_{\mu}\gamma_m\psi_{\nu},$$

where $R_{\mu b,m} = e^{\nu}_{b}R_{\mu\nu,m}$. Supercovariant is defined as an objects without $\partial\epsilon$ terms. It is easily can see that the spin connection, indeed, is supercovariant.

In Einstein-Cartan theory, there is well-known symmetry in the Hilbert action

$$\int d^4xeR(e,\omega(e)+\tau) = \int d^4xe[R(e,\omega(e)) - \tau_{\mu\nu\rho}\tau^{\rho\nu\mu} + (\tau^{\lambda}_{\lambda\mu})^2]$$

under

$$\omega^{ab}_{\mu} \to \omega^{ab}_{\mu} + \tau^{ab}_{\mu}.$$

with $\tau^{ab}_{\mu} = -\tau^{ba}_{\mu}$ In supergravity, we have a similar identity:

$$\mathcal{L}^2(e,\omega(e,\psi)+\tau) + \mathcal{L}^{3/2}(e,\omega(e,\psi)+\tau)$$

$$= \mathcal{L}^2(e,\omega(e,\psi)) + \mathcal{L}^{3/2}(e,\omega(e,\psi)) + \frac{1}{2k^2}(\tau_{\mu\nu\rho}\tau^{\rho\nu\mu} + (\tau^{\lambda}_{\lambda\mu})^2) + \cdots$$

where $\cdots$ are total derivatives. We can choose particularly

$$\tau^{ab}_{\mu} = \omega^{ab}_{\mu}(e,\psi).$$

With this choice, the hilbert action will vanishes. Morever the spin connection will disappear in the gravitino action. Now if we use the equation $R_{\mu\nu,m} = -\partial_{mu}e_{m\nu} + \partial_{nu}e_{m\mu} + \frac{k^2}{2}\bar{\psi}_{\mu}\gamma_m\psi_{\nu}$ which we derived earlier, we can rewrite the action of supergravity without the curvature R but with the torsion terms

$$I = \int d^4x[-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu}\gamma_5\gamma_{\nu}\partial_{\rho}\psi_{\sigma} - \frac{1}{8}R^2_{\mu\nu a} - R_{\mu\nu a}R^{a\nu\mu} + \frac{1}{2}R^2_{\nu\lambda\lambda}]$$

where

$$R_{\mu\nu a} = -\partial_\mu e_{a\nu} + \partial_\nu e_{a\mu} + \frac{k^2}{2}\bar{\psi}_\mu \gamma_a \psi_\nu.$$

3.4 The 1.5 and 2.0 order formalism and Gauge symmetries

From the previous sections, we have seen that the spin connection ω_μ^{ab} serves as the gauge field for the Local Lorentz transfromations. But it is only an auxiliary field and hence it can be eliminated by solving algebraically its equation of motion. This formulation with the spin connection as an auxiliary field is called the first order formulation. And the one which uses the vielbein and the gravitino at the very beginning is called second order formulation. And finally we have a mixed case, which is called the 1.5 formalism.

Let us denote ψ_μ^α as a complex 2-component Weyl spinor field , where α are Weryl spinor indices. Its complex conjugate is denoted by $\bar{\psi}_\mu^{\dot{\alpha}}$. In the first order formulation, the Lagrangian is a function of the vielbein, gravitino, spin connection and their derivatives

$$\mathcal{L} = \frac{1}{2} e E_b^\mu E_a^\nu R_{\mu\nu}^{ab} + 2(\bigtriangledown_\mu \psi_\nu \sigma_\rho \bar{\psi}_\sigma + \psi_\sigma + \psi_\sigma \sigma_\rho \bigtriangledown_\mu \bar{\psi}_\nu)\epsilon^{\mu\nu\rho\sigma}$$

where we have denoted the inverse of the vielbein by E_a^ν, and

$$\bigtriangledown_\mu \psi_\nu^\alpha = \partial_\mu \psi_\nu^\alpha - \frac{1}{2}\omega_\mu^{ab}(\psi_\nu \sigma_{ab})^\alpha,$$

$$\bigtriangledown_\mu \bar{\psi}_\nu^{\dot{\alpha}} = \partial_\mu \bar{\psi}_\nu^{\dot{\alpha}} - \frac{1}{2}\omega_\mu^{ab}(\bar{\sigma}_{ab}\bar{\psi}_\nu)^{\dot{\alpha}}.$$

To determine ω_μ^{ab}, we just vary it in the largrangian above. And one can get

$$\omega_\mu^{ab} = E^{\nu a}\partial_{[\mu}e_{v]}^b - E^{\nu b}\partial_{[\mu}e_{v]}^a - e_{\mu c}E^{\nu a}E^{\rho b}\partial_{[\nu}e_{\rho]}^c + 2i(\psi_\mu \sigma^{[a}\bar{\psi}^{b]} + \psi^{[a}\sigma^{b]}\bar{\psi}_\mu + \psi^{[a}\sigma_\mu \bar{\psi}^{b]}).$$

Once we substitute the above expression of ω_μ^{ab} to the Largrangian in the very beginning, then we have the second order formulation. To see how we get the 1.5 order formulation, we just use a trick to simplify the variation of second order action as it is derived from the first order. The argument is as follows: Suppose we have a Largrangian $\mathcal{L}(\phi, H)$ which involves fields ϕ^i and H^A. And the H^A have the solution $H^A = H^A(\phi)$ from the equation of motion. By considering the second order Largrangian $\mathcal{L}(\phi, H(\phi))$, we can vary it with the fields ϕ^i. Then we have

$$\delta\mathcal{L}(\phi, H(\phi)) \sim [\delta\phi^i \frac{\partial\mathcal{L}(\phi, H)}{\partial\phi^i} + \delta H^A(\phi)\frac{\partial\mathcal{L}(\phi, H)}{\partial H^A}]_{H=H(\phi)} = [\delta\phi^i \frac{\partial\mathcal{L}(\phi, H)}{\partial\phi^i}]_{H=H(\phi)}.$$

where $\mathcal{L}(\phi, H)$ is the first order Lagrangian and $\sim$ deonotes equality up to a total divergence. The term $[\frac{\partial\mathcal{L}(\phi,H)}{\partial H^A}]_{H=H(\phi)} = 0$ because $H^A(\phi)$ solve the equation of motion.

Finally we can show that the Largrangian $\mathcal{L}$ under 1.5 order formalism is invariance under the following spacetime diffeomorphisms, local Lorentz transformations and local SUSY

$$\delta_{Diffeo} e^a_\mu = \xi^\nu \partial_\nu e^a_\mu + \partial_\mu \xi^\nu e^a_\nu,$$

$$\delta_{Diffeo} \psi_\mu = \xi^\nu \partial_\nu \psi_\mu + \partial_\mu \xi^\nu \psi_\nu,$$

$$\delta_{Diffeo} \bar{\psi}_\mu = \xi^\nu \partial_\nu e^a_\mu + \partial_\mu \xi^\nu \bar{\psi}_\nu,$$

$$\delta_{Lorentz} e^a_\mu = \xi^a_b e^b_\mu,$$

$$\delta_{Lorentz} \psi^\alpha_\mu = \frac{1}{2}\xi^{ab}(\psi_\mu \sigma_{ab})^\alpha,$$

$$\delta_{Lorentz} \bar{\psi}^{\dot{\alpha}}_\mu = -\frac{1}{2}\xi^{ab}(\bar{\sigma}_{ab}\bar{\psi}_\mu)^{\dot{\alpha}},$$

$$\delta_{Sussy} e^a_\mu = 2i\xi\sigma^a\bar{\psi}_\mu - 2i\psi_\mu\sigma^a\bar{\xi},$$

$$\delta_{Sussy} \psi^\alpha_\mu = \bigtriangledown_\mu \xi^\alpha,$$

$$\delta_{Sussy} \bar{\psi}^{\dot{\alpha}}_\mu = \bigtriangledown_\mu \bar{\xi}^{\dot{\alpha}}.$$

For the proof, please find it in [14] and [18].

Chapter 4

Extended supergravities

In this chapter we will take a look at the extended supergravities [14] which contain more than one gravitino. This is called N-extended supergravities because N represents the number of real gravitinos. For example when $N = 0$ and $N = 1$, we have general relativity and simple gravity, which we have discussed in the last chapter, respectively. There are only eight viable supergravities, $1 \geq N \geq 8$. It is because for $N > 8$, one would have particles with spins larger then two and also more than one graviton. And it is well-known that the free field action for spin 5/2 is unique and there is no consistence when it is coupled to either gravity or to any simpel matter field. Hence we stop at spin 2, and supergravity at $N = 8$.

4.1 The N = 2 model

So far, we have seen the couplings of chiral and vector multiplets to minmimal $N = 1$ supergravity. But there is another $N = 1$ multiplet that one may consider. It is so-called gravitino multiplet which consists of a spin 3/2 field ψ_μ and a spin 1 field A_μ. Because of the addition of this massless gravitino, we have to introduce another local symmetry. Therefore, coupling the gravitino multiplet to the $N = 1$ supergravity multiplet gives rise to a theory, which has two local symmetries and hence it is referred to as $N = 2$ supergravity. Thus $N = 2$ supergravity comprises one spin 2 field, two spin 3/2 fields and a spin 1 field. And this theory can be extended even further by coupling more gravitino multiplets and arrives at theories with larger extended symmetry. The particle content of the most common multiplets with extended symmetry can be summarised at the following table In

λ	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2	
$N=2$	0	0	0	0	0	0	1	2	1	sugra
	0	0	0	0	1	2	1	0	0	vector
	0	0	0	1	2	1	0	0	0	hyper
$N=4$	0	0	0	0	1	4	6	4	1	sugra
	0	0	1	4	6	4	1	0	0	vector
$N=8$	1	8	28	56	70	56	28	8	1	sugra

Table 4.1: The various multiplets for $N = 2, 4$ and 8 supersymmetry. λ denotes the helicity of the states.

$N = 2$ model, it has a large symmetry $O(2)$ which rotates the two gravitinos into each other. This model can unify eletromagnestism with gravity. The action is

$$\mathcal{L} = -\frac{e}{2}R(e,\omega) - \frac{e}{2}\bar{\psi}^i_\mu \Gamma^{\mu\rho\sigma} D_p(\omega)\psi^i_\sigma - \frac{e}{4}F^2_{\mu\gamma} + \frac{k}{4\sqrt{2}}\bar{\psi}^i_\mu[e(F^{\mu\nu} + \hat{F}^{\mu\nu}) + \frac{1}{2}\gamma_5(\tilde{F}^{\mu\nu} + \tilde{\hat{F}}^{\mu\nu})]\psi^j_\nu \epsilon^{ij}$$

and it is invariant under general coordinate, local Lorentz and Maxwell transformations

$$\delta A_\mu = \partial_\mu \Lambda$$

as well as the following two local supersymmetries

$$\delta e^m_\mu = \frac{k}{2}\bar{\epsilon}^i \gamma^m \psi^i_\mu,$$

$$\delta A_\mu = \frac{k}{\sqrt{2}}\bar{\epsilon}^i \psi^j_\mu \epsilon^{ij}$$

$$\delta\psi^i_\mu = D_\mu(\omega)\epsilon^i + \frac{k}{2\sqrt{2}}\epsilon^{ij}(\hat{F}_{\mu\lambda}\gamma^\lambda + \frac{1}{2}e\tilde{\hat{F}}_{\mu\lambda}\gamma^\lambda\gamma_5)\epsilon^j.$$

where

$$\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

and

$$\hat{F}_{\mu\nu} = (\partial_\mu A_\nu - \frac{k}{2\sqrt{2}}\bar{\psi}^i_\mu \psi^j_\nu \epsilon^{ij}) - (\mu \leftrightarrow \nu).$$

And each gravitino gauges one local supersymmetry as can be seen from $\delta\psi^i_\mu = \partial_\mu \epsilon^i +$ more.

The maximal supersymmetry group of the S-matrix of supersymmetry algebras containing the Poincare algebra has been shown that it is $U(2)$. But in fact the field theory has this symmetry only on-shell. For Off-shell, it is $SU(2)$ which remains. These symmetries are

$$\delta\psi^L_\mu = i\omega \cdot \tau \psi^L_\mu$$

$$\delta\psi^R_\mu = -i\omega \cdot \tau \psi^R_\mu$$

where

$$\psi_\mu = (\psi^1_\mu, \psi^2_\mu)^T,$$

$$\psi^L_\mu = \frac{1}{2}(1+\gamma_5)\psi_\mu$$

$$\psi^R_\mu = \frac{1}{2}(1-\gamma_5)\psi_\mu.$$

And For $U(1)$,

$$\delta\psi_\mu = -i\gamma_5\psi_\mu,$$

$$\delta\hat{F}_{\mu\nu} = ie\epsilon_{\mu\nu\rho\sigma}\hat{F}^{\rho\sigma}.$$

Hence the $SU(2)$ rotates ψ^L as (2) and ψ^R as $(\bar{2})$. And the $U(1)$ is a combined chirality-duality transformations.

It is well-known that when we couple the photon minimally to the femions, we need a cosmological constant and a mass like term in the action. So in the $N=2$ model, we also add a cosmological constant to the action and get a further modification of the Noether method. The extra terms are

$$\mathcal{L}(cosm) = 6eg^2 + 2e\bar{\psi}^i_\mu \sigma^{\mu\nu}\psi^i_\nu - \frac{1}{2}\bar{\psi}^i_\mu \Gamma^{\mu\rho\sigma}(D_\rho\psi^i_\sigma + g\epsilon^{ik}A_\rho\psi^k_\sigma),$$

$$\delta\psi^i_\mu = D_\mu(\omega(e,\psi))\epsilon^i + g\gamma_\mu\epsilon^i + g\epsilon^{ik}A_\mu\epsilon^k$$

where g is a dimensionless gauge coupling.

4.2 The N = 3 model

Now we take a look at the $N = 3$ model. In this model, the Abelian photon gauge invariance turns into a non-Abelian Yang-Mills invariance in de-Sitter space. As in the $N = 2$ model, we also need a cosmological constant and a mass term when coupling the triplet of photons to the gravitinos. The action in this model is

$$\mathcal{L} = \mathcal{L}(Kinetic) + \frac{1}{2}(\mathcal{L}^{Noether}_{bare} + \mathcal{L}^{Noether}_{superconv}) + \mathcal{L}(cosm)$$

where $\mathcal{L}(Kinetic)$ is the kinetic terms for $e^m_\mu, \psi^i_\mu, A^i_\mu$ and λ with $i = 1 \ldots 3$ and

$$\mathcal{L}(cosm) = 6eg^2 + 2e\bar{\psi}^i_\mu \sigma^{\mu\nu}\psi^i_\nu - \frac{1}{2}\bar{\psi}^i_\mu \Gamma^{\mu\rho\sigma}(D_\rho \psi^i_\sigma + g\epsilon^{ijk}A^j_\rho \psi^k_\sigma).$$

The bare Noether coupling is a sum of the Noether couplings of the $(\frac{3}{2}, 1)$ and $(1, \frac{1}{2})$ systems to the $(2, \frac{3}{2})$ system. And it is defined as

$$\mathcal{L}^{Noether}_{bare} = -\frac{1}{2\sqrt{2}}\bar{\psi}^i_\mu(eF^{\mu\nu,j} + \frac{1}{2}\gamma_5 \tilde{F}^{\mu\nu,j})\psi^k_\nu \epsilon^{ijk} + \frac{1}{2}(\bar{\psi}^i_\mu \sigma^{\alpha\beta}\gamma^\mu \lambda)(F^i_{\alpha\beta}).$$

Finally the supercovariantised Noether coupling follows from the local supersymmetry transformation rules

$$\delta e^m_\mu = \frac{1}{2}\bar{\epsilon}^i \gamma^m \psi^i_\mu,$$

$$\delta\lambda = \frac{1}{2}(\sigma^{\mu\nu}\epsilon^i)(\hat{F}^i_{\mu\nu}),$$

$$\delta A^i_\mu = \frac{1}{\sqrt{2}}\epsilon^{ijk}\bar{\epsilon}^j \psi^k_\mu - \frac{1}{2}\bar{\epsilon}^i \gamma_\mu \lambda,$$

$$\delta\psi^i_\mu = D_\mu(\omega(e,\psi,\lambda))\epsilon^i + \frac{1}{2\sqrt{2}}\epsilon^{ijk}(\sigma^{\rho\sigma}\gamma_\mu \epsilon^k)(\hat{F}^j_{\rho\sigma}) + \frac{1}{4\sqrt{2}}\epsilon^{ijk}[(\bar{\psi}^j_\mu \gamma_\rho \lambda)(\gamma^\rho \epsilon^k) + (\bar{\psi}^j_\mu \gamma_\rho \gamma_5 \lambda)(\gamma_5 \gamma^\rho \epsilon^k)]$$
$$+ \frac{1}{8}(\bar{\lambda}\gamma_5 \gamma^\rho \lambda)(\gamma_\rho \gamma_\mu \gamma_5 \epsilon^i) + g\epsilon^{ijk}A^j_\mu \epsilon^k + g\gamma_\mu \epsilon^i.$$

$\delta\psi_\mu$ is no longer supercovariant. The noncovariant terms involve the spin 1/2 fields. This means that there must be some extra terms in the commutator of two local supersymmetry transformatins. The first one is an extra local Lorentz transformation with parameter.

$$(2\sqrt{2})^{-1}\epsilon^{ijk}(\bar{\epsilon}^i_2 \epsilon^k_1 \hat{F}^j_{mn} + \frac{e}{2}\bar{\epsilon}^i_2 \gamma_5 \epsilon^k_1 \hat{\tilde{F}}^j_{mn})$$

and an extra local supersymmetry transformation with parameter

$$(2\sqrt{2})^{-1}\epsilon^{ijk}(\bar{\epsilon}_2^j\epsilon_1^k\lambda - \bar{\epsilon}_2^j\gamma_5\epsilon_1^k\gamma_5\lambda)$$

and finally an $O(3)$ Yang-Mills gauge transformation with parameter

$$\Lambda^i = (2\sqrt{2})^{-1}\epsilon^{ijk}(\bar{\epsilon}_2^j\epsilon_1^k) - \frac{1}{2}\bar{\epsilon}_2^j\gamma^\mu\epsilon_1^j A_\mu^i,$$

$$\delta A_\mu = \partial_\mu\Lambda^i + g\epsilon^{ijk}A_\mu^j\Lambda^k,$$

$$\delta e_\mu^m = 0,$$

$$\delta\psi_\mu^i = g\epsilon^{ijk}\psi_\mu^j\Lambda^k,$$

$$\delta\lambda = 0.$$

For $g = 0$ on-shell, there is a $U(3)$ global invariance. However there is only $O(3)$ invariance left when it is off-shell. This is due to the fact that all other symmetries involve not only chiral transformations but also duality transfromations as well. The precise rule follows from the truncation from the $U(4)$ group of the $SO(4)$ version of the $N = 4$ model. The truncation here means letting certain fields equal to zero. For $g \neq 0$, the Yang-Mills coupling includes the bare fields A_μ^i and the mass term which break the chiral invariance, and hence it is left with $O(3)$ invariance.

In $N = 3$ model, we can actually obtain other theories by consistence truncation. It means that while we are performing the truncation, we also require that their variations also vanish. For example, when we let $A_\mu^1 = A_\mu^2 = \psi_\mu^3 = \lambda = 0$ in the $N = 3$ model, it will reduce to the $N = 2$ model. However, if we, instead, let $A_\mu^2 = A_\mu^3 = \psi_\mu^2 = \psi_\mu^3 = 0$, then we will have $(2, 3/2) + (1, 1/2)$ Maxwell-Einstein system.

4.3 The N = 4 model

So far, there are only two $N = 4$ discovered. They are $SO(4)$ model with fields $(e, \psi'^i, V_\mu^{ij}, \lambda'^i, A, B')$ and the SU_4 model with fields $(e, \psi^i, A_\mu^k, B_\mu^k\lambda^i, \phi, B)$. The six fields V_μ^{ij} are all vector fields, but the three A_μ^k are vector fields while the B_μ^k are axial vector fields. for $SU(4)$ model, one can actually obtain it by reduction of the $N = 1$ model in $d = 10$ dimensions.

It is interesting to see that these two theroies are equivalent at the classical level by using a transformation law of the field which turns the one action into the other. For more details, please see [14].

Chapter 5

Conclusions

We have seen that supergravity plays such an important role in modern physics as it predicts the existence of a boson particle known as the graviton and its femionic superpartner, the gravitino. Its presence is likely to improve the quantum behavior of the theory, particularly interesting in the context of gravity, a notoriously non-renormalizable theory. In this dissertation, the basic idea of the supergravity has been demonstrated through both the simple model with only one coordinate t and also in $N = 1$ $4 - d$ model. Some extended supergravity models have also been included. To extended all these simple models is important. For example, supergravity appears as the low energy effective action for fundamental theories such as string theories, which generically live in higher dimensions. So one of the suggestions of future works is to investigate other extensions of the supergravity model which are not covered in this dissertation.

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